Seminar Talk: Admissible lattices and their discrepancy

Jan Zimmermann

Admissible lattices are certain lattices in the Euclidean space $\mathbb{R}^d$ which satisfy some geometric admissibility condition. They have small discrepancy with respect to axes-parallel rectangles, in the sense that the discrepancy only grows logarithmically in the measure of the rectangle.

In the first part of this talk, we will summarize some basic properties of and construction methods for admissible lattices, and state the theorem about their discrepancy. In the second part, we will give a proof some parts of that theorem. We also mention connections to the cubature of functions with bounded mixed derivatives.

Notation and basic definitions:

- We always let $d \in \mathbb{N}$.
- For $q \in (0, \infty)^d$ we write $[0, q) = [0, q_1) \times \cdots \times [0, q_d)$.
- If $x \in \mathbb{R}^d$ and $q \in (0, \infty)^d$, we consider axes-parallel rectangles of the form $Q = x + [0, q)$.
- We write μ for the Lebesuge-measure and $\#$ for the cardinality of a set.
- Given an invertible matrix $A \in \mathbb{R}^{d \times d}$, we call the set $\Gamma = A\mathbb{Z}^d$ a lattice. The dual lattice of Γ is defined as

$$\Gamma^* = A^{-T}\Gamma = \{\gamma^* \in \mathbb{R}^d \mid \forall \gamma \in \Gamma : \ \langle \gamma, \gamma^* \rangle \in \mathbb{Z}\}.$$

  The determinant of Γ is defined as $\det(\Gamma) = |\det(A)|$.

The central question:

Given a rectangle $Q = x + [0, q)$ and a lattice $\Gamma \subset \mathbb{R}^d$, how large is the discrepancy

$$\Delta_\Gamma(Q) := \mu(Q) - \det(\Gamma)\#(\Gamma \cap Q) \quad ?$$

Some examples for 'bad' lattices:

- If $\Gamma = \mathbb{Z}^d$ and $Q = x + [0, a)^d$ with $a > 0$, then there is some constant $C > 0$ such that

$$|\Delta_\Gamma(Q)| \leq C(a^{d-1} + 1).$$

- For $\Gamma = \mathbb{Z}^2$ and $Q = x + [0,a) \times [0,b)$ with $a, b > 0$ and $x \in \mathbb{R}^2$, there is some constant $C > 0$ such that
$$|\Delta_\Gamma(Q)| \leq C(a + b + 1).$$
Thus, the discrepancy does not just depend on the measure $\mu(Q) = ab$, but on the individual side lengths, i.e. the eccentricity of the rectangle.

For admissible lattices, we have lower asymptotical bounds for $|\Delta_\Gamma|$, which *only* depend on the measure $\mu(Q)$ of the axes-parallel rectangle!

Definition 1. For $\lambda \in \mathbb{R}^d$, we define the norm

$$\mathrm{Nm}(\lambda) = \prod_{j=1}^{d} \lambda_j$$

(This is not a norm in the sense of vector spaces; the term 'norm' comes from algebraic number theory). Note that for $q \in (0, \infty)^d$ we have $\mathrm{Nm}(q) = \mu([0, q))$.

For a lattice $\Gamma \subset \mathbb{R}^d$, we define

$$\mathrm{Nm}(\Gamma) = \inf_{\gamma \in \Gamma \setminus \{0\}} |\mathrm{Nm}(\gamma)|.$$

We call Γ *admissible* if $\mathrm{Nm}(\Gamma) > 0$.

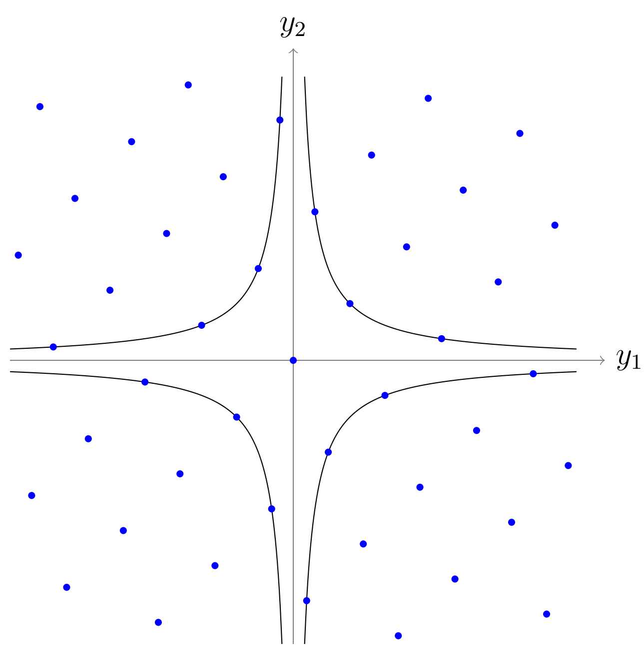

Figure 1: The admissible golden ratio lattice (defined further below) and the hyperbolas $|y_1 y_2| = 1$.

From now on, Γ is always assumed to be an admissible lattice.

Some basic properties of admissible lattices:

- If $Q = x + [0, q)$ with $\mu(Q) \leq \mathrm{Nm}(\Gamma)$, then $\#(\Gamma \cap Q) \leq 1$.

  Proof by contradiction: Assume that $\gamma, \gamma' \in \Gamma \cap Q$ are two distinct points. Then $\gamma_j - \gamma'_j < q_j$ for all $j = 1, \ldots, d$, so

$$\mathrm{Nm}(\Gamma) \leq |\,\mathrm{Nm}(\gamma - \gamma')| < \mathrm{Nm}(q) = \mu(Q),$$

  a contradiction.

- In more generality, if $\mu(Q)$ is arbitrary, we have

$$\#(\Gamma \cap Q) \leq \left\lceil \frac{\mu(Q)}{\mathrm{Nm}(\Gamma)} \right\rceil \leq 1 + \mathrm{Nm}(\Gamma)^{-1}\mu(Q).$$

  Proof: We cover Q by a union of axes-parallel rectangles $Q_1, \ldots, Q_N$, $N \in \mathbb{N}$, which all have measure $\mu(Q_j) = \mathrm{Nm}(\Gamma)$. We can do this with $N = \left\lceil \frac{\mu(Q)}{\mathrm{Nm}(\Gamma)} \right\rceil$ rectangles, while each rectangle contains at most one point from Γ.

- If $\lambda \in (0, \infty)^d$, then

$$\det(\mathrm{diag}(\lambda)\Gamma) = \det(\mathrm{diag}(\lambda)) \det(\Gamma) = \mathrm{Nm}(\lambda) \det(\Gamma).$$

  Similarly,

$$\mathrm{Nm}(\mathrm{diag}(\lambda)\Gamma) = \mathrm{Nm}(\lambda)\,\mathrm{Nm}(\Gamma).$$

  Thus, admissibility as property is invariant under dilations. Moreover, if $\mathrm{Nm}(\lambda) = 1$ (i.e. $\mathrm{diag}(\lambda)$ is a unimodular dilation), then the determinant and norm of the lattice remain unchanged.

- If $Q = [0, q)$, there is some $\lambda \in (0, \infty)^d$ with $\mathrm{Nm}(\lambda) = 1$ and $a > 0$ such that

$$\mathrm{diag}(\lambda) Q = [0, a)^d,$$

  and in particular $\mu(Q) = a^d$. Then

$$\begin{aligned}\Delta_\Gamma(Q) &= \mu(Q) - \#(\Gamma \cap Q) \\ &= \mu([0, a)^d) - \#(\mathrm{diag}(\lambda)\Gamma \cap [0, a)^d) \\ &= \Delta_{\mathrm{diag}(\lambda)\Gamma}([0, a)^d),\end{aligned}$$

  where $\mathrm{Nm}(\mathrm{diag}(\lambda)\Gamma) = \mathrm{Nm}(\Gamma)$ and $\det(\mathrm{diag}(\lambda)\Gamma) = \det(\Gamma)$. Thus, any upper bound for $|\Delta_\Gamma([0, a)^d)|$ that only depends on $\mathrm{Nm}(\Gamma)$ and $\det(\Gamma)$ immediately can be extended to all axes-parallel rectangles!

- Γ is admissible $\quad\Longleftrightarrow\quad$ Γ^* is admissible.

Construction methods for admissible lattices:

- Let $p(z) = z^d + a_{d-1}z^{d-1} + \cdots + a_0 \in \mathbb{Z}[z]$ be an irreducible polynomial with integer coefficients and d distinct, irrational roots $\alpha_1, \ldots, \alpha_d \in \mathbb{R}$. Then

$$\Gamma = \begin{pmatrix} 1 & \alpha_1 & \alpha_1^2 & \cdots & \alpha_1^{d-1} \\ 1 & \alpha_2 & \alpha_2^2 & \cdots & \alpha_2^{d-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \alpha_d & \alpha_d^2 & \cdots & \alpha_d^{d-1} \end{pmatrix} \mathbb{Z}^d$$

  is admissible with $\mathrm{Nm}(\Gamma) = 1$ (and $\det(\Gamma)^2 =$ Discriminant of p). The proof uses Vieta's formulas and follows the same idea as the next example.

- The golden ratio lattice: Consider $p(z) = z^2 - z - 1$, which has has the roots $\varphi = (1+\sqrt{5})/2$ and $-1/\varphi = (1-\sqrt{5})/2$. This gives the lattice

$$\Gamma = \begin{pmatrix} 1 & \varphi \\ 1 & -1/\varphi \end{pmatrix} \mathbb{Z}^2,$$

  which is admissible with $\mathrm{Nm}(\Gamma) = 1$ and $\det(\Gamma) = \sqrt{5}$.

  Proof: For $(k,l) \in \mathbb{Z}^2 \backslash \{(0,0)\}$, we have

$$\mathrm{Nm}\left(\begin{pmatrix} 1 & \varphi \\ 1 & -1/\varphi \end{pmatrix} \begin{pmatrix} k \\ l \end{pmatrix} \right) = (k + l\varphi)(k - l/\varphi) = k^2 - l^2 + kl \in \mathbb{Z}.$$

  Also $(k+l\varphi)(k-l/\varphi) \neq 0$, as each factor $k+l\varphi$ and $k-l/\varphi$ can only vanish if $k = l = 0$. Thus we have $\mathrm{Nm}(\gamma) \in \mathbb{Z}\backslash\{0\}$ for all $\gamma \in \Gamma\backslash\{0\}$, which implies $\mathrm{Nm}(\Gamma) \geq 1$.

  On the other hand, $(1,1)^T \in \Gamma$, so $\mathrm{Nm}(\Gamma) \leq 1$.

- In the two dimensional case, it can be shown that for $\alpha, \beta \in \mathbb{R}$ we have

$$\Gamma = \begin{pmatrix} 1 & \alpha \\ 1 & \beta \end{pmatrix} \mathbb{Z}^2 \quad \text{is admissible} \quad \iff \quad \alpha, \beta \text{ are distinct, badly approximable numbers.}$$

  A number is called *badly approximable* if its continued fraction expansion is bounded.

  This leads to the following characterization of admissible lattices: $\Gamma \subset \mathbb{R}^2$ is admissible if and only if there are distinct, badly approximable $\alpha, \beta \in \mathbb{R}$ and some $\lambda \in (\mathbb{R}\backslash\{0\})^2$ such that

$$\Gamma = \mathrm{diag}(\lambda) \begin{pmatrix} 1 & \alpha \\ 1 & \beta \end{pmatrix} \mathbb{Z}^2.$$

  In higher dimensions $d \geq 3$, there is no known characterization of admissible lattices similar to this.

Upper bounds for the discrepancy:

Theorem 2 ([1]). *Let Γ be admissible with fundamental domain $\mathcal{F}$. Let $Q = [0, q)$ for $q \in (0, \infty)^d$. Then for $p \in (0, \infty)$, there exists a constant $C = C(p, d, \det(\Gamma), \mathrm{Nm}(\Gamma)) > 0$ such that we have the following upper bounds for the discrepancy:*

- *L^p-discrepancy for $p \in (0, \infty)$:*

$$\left(\int_{\mathcal{F}} |\Delta_\Gamma(x + Q)|^p \, dx \right)^{1/p} \leq C \log(2 + \mu(Q))^{\frac{d-1}{2}}.$$

- *L^∞-discrepancy:*

$$\sup_{x \in \mathcal{F}} |\Delta_\Gamma(x + Q)| \leq C \log(2 + \mu(Q))^{d-1}.$$

Relation to cubature formulas:

For some continuous function $f : [0, 1]^d \to \mathbb{C}$, we define the cubature error of f with respect to Γ by

$$e(f, \Gamma) = \int_{[0,1]^d} f(x) \, dx - \det(\Gamma) \sum_{\gamma \in \Gamma \cap [0,1]^d} f(\gamma).$$

Theorem 3 ([1]). *Let $\Gamma \subset \mathbb{R}^d$ be an admissible lattice and $f \in C([0, 1]^d)$ supported* inside *of $[0, 1]^d$, i.e. f vanishes on the boundary of $[0, 1]^d$. Assume that the mixed partial derivative $\partial_{1...d} f$ exists. Let $p \in [1, \infty]$. Then there is some constant $C = C(p, d, \det(\Gamma), \mathrm{Nm}(\Gamma))$ so that:*

- *For $p \in (1, \infty]$ and all $a > 0$ we have*

$$|e(f, a\Gamma)| \leq C \|\partial_{1...d} f\|_p \, a^d \log(2 + a^{-1})^{\frac{d-1}{2}}.$$

- *For $p = 1$ and all $a > 0$ we have*

$$|e(f, a\Gamma)| \leq C \|\partial_{1...d} f\|_1 \, a^d \log(2 + a^{-1})^{d-1}.$$

There are many upper bounds $|e(f, a\Gamma)|$ similar in spirit to the above ones, where the norm $\|\partial_{1...d} f\|_p$ is replaced with a different notion of *mixed smoothness* [2, section 6.7][3]:

- Often, the norms of the lower-order mixed partial derivatives are also included: In multi-index notation, these are the derivatives $\partial^\alpha f$, where $\alpha \in \mathbb{N}_0^d$ with $\|\alpha\|_\infty \leq 1$.
- Stronger smoothness assumptions lead to smaller asymptotical bounds. Typically, one considers the mixed partial derivatives $\partial^\alpha f$ for $\alpha \in \mathbb{N}_0^d$, $\|\alpha\|_\infty \leq r$ where $r \geq 1$ is the order of mixed smoothness.
- According Sobolev-, Besov- and Triebel-Lizorkin-space constructions do exist as well and lead to upper cubature bounds [4].

The above theorem is interesting for the following reason: If Γ is a non-admissible lattice and we want to achieve an asymptotical bound $|e(f, a\Gamma)| \in \mathcal{O}(a^d)$ for $a \to 0$, then we have to assume f to be d-times differentiable such that $\|\partial^\alpha f\|_p < \infty$ for all $\alpha \in \mathbb{N}_0^d$ with $\|\alpha\|_1 \leq d$. That is, we need that all partial derivatives up to order d exist, including derivatives with order greater than 1 in individual directions (like $\partial_{1\ldots1} f$). On the other hand, Skriganovs theorem gives us an upper bound for $|e(f, a\Gamma)|$ for admissible Γ which is almost as good as $\mathcal{O}(a^d)$ but only requires the derivatives $\partial^\alpha f$ with $\|\alpha\|_\infty \leq 1$ to exist and have finite L^p-norm.

References

[1] M. M. Skriganov. “Constructions of uniform distributions in terms of geometry of numbers”. In: *Algebra i Analiz* 6.3 (1994), pp. 200–230.

[2] V. Temlyakov. *Multivariate approximation.* Vol. 32. Cambridge Monographs on Applied and Computational Mathematics. Cambridge University Press, Cambridge, 2018. DOI: 10.1017/9781108689687.

[3] Mario Ullrich. “On “Upper error bounds for quadrature formulas on function classes” by K. K. Frolov”. In: *Monte Carlo and quasi-Monte Carlo methods.* Vol. 163. Springer Proc. Math. Stat. Springer, [Cham], 2016, pp. 571–582. DOI: 10.1007/978-3-319-33507-0_31.

[4] Mario Ullrich and Tino Ullrich. “The role of Frolov’s cubature formula for functions with bounded mixed derivative”. In: *SIAM J. Numer. Anal.* 54.2 (2016), pp. 969–993. DOI: 10.1137/15M1014814.