L^2-DISCREPANCIES FOR LATTICES AND BEYOND

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Structure of the talk:

(1) Vague introduction

(2) Gauss' circle problem

(3) The L^2-discrepancy of a lattice and main theorems

(4) A spectral formula

(5) The bigger picture: Dynamical irregularities of distribution

1. Introduction

We are interested in understanding the statistics of counting functions

$$r \longmapsto \#(P \cap rK)$$

as $r \to +\infty$, where $P \subset \mathbb{R}^d$ is a locally finite subset and $K \subset \mathbb{R}^d$ is, say, compact with non-empty interior and $rK = \{rx \,|\, x \in K\}$. By "statistics" we mean with respect to some probability governed by varying P in some family of locally finite sets, e.g. the set itself(often hard) or its translations(often more tractable). When the point set P is sufficiently well-distributed, one often "expects" the point set P to have finite asymptotic density $\rho = \rho(P) > 0$ in the sense that

$$\lim_{r \to +\infty} \frac{\#(P \cap rK)}{|rK|} = \rho \in (0, +\infty)\,.$$

Given this situation one is inclined to understand the rate of growth of the *discrepancy*

$$\mathcal{D}_{rK}(P) = \#(P \cap rK) - \rho|rK| = o(r^d)\,. \tag{1.1}$$

2. The Gauss circle problem

The classical Gauss circle problem (in arbitrary dimension) addresses the situation where $P = \mathbb{Z}^d$ and $K = B_1(0)$. Here the counting function also has number theoretic significance,

$$\#(\mathbb{Z}^d \cap B_r(0)) = \sum_{m=1}^{\lfloor r \rfloor^2} r_d(m)$$

where

$$r_d(m) = \#\{\gamma \in \mathbb{Z}^d \,|\, \gamma_1^2 + \cdots + \gamma_d^2 = m\}$$

is the number of ways to represent a positive integer m as the sum of d squared integers.

Lemma 2.1 (Gauss' geometric estimate)**.** *There is an $r_o(d) > 0$ such that*

$$|B_r(0)| - |B_r(0) \backslash B_{r-\sqrt{d}}(0)| \le \#(\mathbb{Z}^d \cap B_r(0)) \le |B_r(0)| + |B_{r+\sqrt{d}}(0) \backslash B_r(0)|$$

for every $r \ge r_o(d)$.

Corollary 2.2. *The integer lattice $\mathbb{Z}^d$ has asymptotic density* 1 *in $\mathbb{R}^d$ and the discrepancy over balls satisfies*

$$|\mathcal{D}_{B_r(0)}(\mathbb{Z}^d)| = O(r^{d-1})\,.$$

With very little effort we have improved on 1.1 for lattices, but the error bound $O(r^{d-1})$ is far from optimal. For the following statements we recall the Hardy-Littlewood Ω-notation,

$$f(r) = \Omega_+(g(r)) \Leftrightarrow \limsup_{r\to+\infty} \frac{f(r)}{g(r)} > 0\,, \quad f(r) = \Omega_-(g(r)) \Leftrightarrow \liminf_{r\to+\infty} \frac{f(r)}{g(r)} < 0$$

and $f(r) = \Omega_\pm(g(r))$ if both hold. The expected result is the following.

Conjecture: $\mathcal{D}_{B_r(0)}(\mathbb{Z}^d) = O(r^{\alpha(d)+\varepsilon})$ and $\mathcal{D}_{B_r(0)}(\mathbb{Z}^d) = \Omega_+(r^{\alpha(d)})$ where

$$\alpha(d) = \begin{cases} 1/2 & \text{if } d = 2 \\ d-2 & \text{if } d \ge 3\,. \end{cases}$$

We mention some known (many that are currently the best) asymptotics of the lattice discrepancy over balls. For a nice survey on the best known results (in '04), see [1].

Theorem 2.3. *The lattice point discrepancy over balls satisfies*

- $\mathcal{D}_{B_r(0)}(\mathbb{Z}^2) = O(r^{\frac{131}{208}+\varepsilon})$ *(Huxley) and* $\mathcal{D}_{B_r(0)}(\mathbb{Z}^2) = \Omega_\pm(r^{\frac{1}{2}})$ *(Hardy)*
- $\mathcal{D}_{B_r(0)}(\mathbb{Z}^3) = O(r^{\frac{21}{16}+\varepsilon})$ *(Heath-Brown) and* $\mathcal{D}_{B_r(0)}(\mathbb{Z}^3) = \Omega_\pm(r\log(r)^{\frac{1}{2}})$ *(Tsang)*
- $\mathcal{D}_{B_r(0)}(\mathbb{Z}^4) = O(r^2\log(r)^{\frac{2}{3}})$ *(Walfisz) and* $\mathcal{D}_{B_r(0)}(\mathbb{Z}^4) = \Omega_\pm(r^2\log_2(r))$ *(Adhikari-Pétermann)*
- $\mathcal{D}_{B_r(0)}(\mathbb{Z}^d) = O(r^{d-2})$ *and* $\mathcal{D}_{B_r(0)}(\mathbb{Z}^d) = \Omega_+(r^{d-2})$ *for all $d \ge 5$. ("well-known")*

These results require *heavy* machinery from different areas of number theory, and just using basic Fourier analysis does not result in these bounds. The difficulty in proving the conjecture is rooted in the wild behavior of the counting functions r_d, especially for $d = 2, 3, 4$.

3. Mean square discrepancy for lattices

Studying "averages" of the lattice point discrepancy $\mathcal{D}_{\mathbb{Z}^d}(B_r(0))$ for large r is often much more susceptible to Fourier analytic techniques. Today we'll consider the *mean square discrepancy* or L^2*-discrepancy*

$$\mathcal{D}_K^{L^2}(P) = \Big(\int_{[0,1)^d} |\mathcal{D}_K(t+P)|^2 dt\Big)^{1/2}$$

with emphasis on the standard lattice and Euclidean balls, given by

$$\begin{aligned}\mathcal{D}_{B_r(0)}^{L^2}(\mathbb{Z}^d) &= \Big(\int_{[0,1)^d} |\mathcal{D}_{B_r(0)}(t+\mathbb{Z}^d)|^2 dt\Big)^{1/2} \\ &= \Big(\int_{[0,1)^d} \Big|\#((t+\mathbb{Z}^d)\cap B_r(0)) - |B_r(0)|\Big|^2 dt\Big)^{1/2}.\end{aligned}$$

A probabilistic interpretation is that $\mathcal{D}_{\mathbb{Z}^d}^{L^2}(B_r(0))$ is the standard deviation of the random variable $N_r : t \mapsto \#((t+\mathbb{Z}^d)\cap B_r(0))$ with respect to the Lebesgue probability measure on the fundamental domain $[0,1)^d$ of the lattice $\mathbb{Z}^d$. Indeed, the expectation of this random variable is

$$\begin{aligned}\mathbb{E}(N_r) &= \int_{[0,1)^d} \#((t+\mathbb{Z}^d)\cap B_r(0))dt = \int_{[0,1)^d} \sum_{\gamma\in\mathbb{Z}^d} \chi_{B_r(0)}(t+\gamma)dt \\ &= \sum_{\gamma\in\mathbb{Z}^d} \int_{[0,1)^d+\gamma} \chi_{B_r(0)}(t)dt = \int_{\mathbb{R}^d} \chi_{B_r(0)}(t)dt = |B_r(0)|,\end{aligned}$$

so the variance is

$$\begin{aligned}\mathrm{Var}(N_r) &= \int_{[0,1)^d} \Big|X_r(t) - \mathbb{E}(X_r)\Big|^2 dt \\ &= \int_{[0,1)^d} \Big|\#((t+\mathbb{Z}^d)\cap B_r(0)) - |B_r(0)|\Big|^2 dt = \mathcal{D}_{B_r(0)}^{L^2}(\mathbb{Z}^d)^2.\end{aligned}$$

Moreover, an interpretation from the point of view of statistical mechanics is that the variance $\mathcal{D}_{B_r(0)}^{L^2}(\mathbb{Z}^d)^2$ records the *spatial fluctuations* of the spatially random "crystal" $t \mapsto t+\mathbb{Z}^d$.

The main results we'll discuss in today's talk are due to Beck and Sobolev-Parnovski respectively in [2, Thm 2A] and [3, Thm 3.1].

Theorem 3.1 (Beck, '87)**.** *Let P be locally finite and K compact and convex in $\mathbb{R}^d$. As $r \to +\infty$, one has*

$$\limsup_{r\to+\infty} \frac{\mathcal{D}^{L^2}_{rK}(P)}{r^{\frac{d-1}{2}}} > 0\,.$$

We'll prove this for $P = \mathbb{Z}^d$, $K = B_1(0)$.

Theorem 3.2 (Sobolev-Parnovski, '99)**.** *As $r \to +\infty$, one has*

$$\liminf_{r\to+\infty} \frac{\mathcal{D}^{L^2}_{B_r(0)}(\mathbb{Z}^d)}{r^{\frac{d-1}{2}}} = 0$$

if and only if $d \equiv 1 \bmod 4$.

In fact, Sobolev-Parnovski prove that if $d \equiv 1 \bmod 4$ then there is a sequence $r_j \to +\infty$ and an absolute constant $C_d > 0$ such that

$$\mathcal{D}^{L^2}_{B_{r_j}(0)}(\mathbb{Z}^d) \le C_d r_j^{\frac{d-1}{2}} \log(r_j)^{\frac{-1+\varepsilon}{2d}}$$

for every $\varepsilon > 0$. Moreover, they prove that for any d there is a $C'_d > 0$ such that

$$\mathcal{D}^{L^2}_{B_r(0)}(\mathbb{Z}^d) \le C'_d r^{\frac{d-1}{2}}\,,$$

so from Theorem 3.2 we see that

$$\mathcal{D}^{L^2}_{B_r(0)}(\mathbb{Z}^d) \asymp r^{\frac{d-1}{2}}$$

if and only if $d \neq 1, 5, 9, 13, \ldots$.

Remark 3.3. When $d = 2$, the exponent $\frac{d-1}{2} = \frac{1}{2}$ matches the conjectured exponent in the Gauss circle problem, so the exact behavior of the counting function $\#(\mathbb{Z}^2 \cap B_r(0))$ is expected to have a slowly increasingly deviating error. This is in contrast to higher dimensions, where it is not expected to hold.

4. A spectral formula

As observed in the survey by Brandolini-Travaglini, the function $t \mapsto \mathcal{D}_K(t+\mathbb{Z}^d)$ is $\mathbb{Z}^d$-periodic by definition and piecewise continuous, so it admits a Fourier series

$$\mathcal{D}_K(t+\mathbb{Z}^d) = \sum_{\gamma\in\mathbb{Z}^d} c_K(\gamma) \mathrm{e}^{2\pi i\langle t,\gamma\rangle}$$

for Lebesgue-almost every t with coefficients given by

$$c_K(\gamma) = \int_{[0,1)^d} \mathcal{D}_K(t+\mathbb{Z}^d)\mathrm{e}^{-2\pi i\langle t,\gamma\rangle} dt$$

$$= \int_{[0,1)^d} \#((t+\mathbb{Z}^d)\cap K)\mathrm{e}^{-2\pi i\langle t,\gamma\rangle}dt - |K|\chi_{\{0\}}(\gamma)\,.$$

Since $t \mapsto \mathrm{e}^{-2\pi i\langle t,\gamma\rangle}$ is $\mathbb{Z}^d$-periodic, a similar computation to that of the expectation $\mathbb{E}(X_r)$ above gives us

$$c_K(\gamma) = \int_{\mathbb{R}^d} \chi_K(t)\mathrm{e}^{-2\pi i\langle t,\gamma\rangle}dt - |K|\chi_{\{0\}}(\gamma) = \widehat{\chi}_K(\gamma)(1-\chi_{\{0\}}(\gamma))\,.$$

Thus Parseval's Theorem yields

$$\begin{aligned}\mathcal{D}_K^{L^2}(\mathbb{Z}^d)^2 &= \int_{[0,1)^d} |\mathcal{D}_K(t+\mathbb{Z}^d)|^2 dt \\ &= \sum_{\gamma\in\mathbb{Z}^d} |c_K(\gamma)|^2 = \sum_{\gamma\in\mathbb{Z}^d\setminus\{0\}} |\widehat{\chi}_K(\gamma)|^2\,.\end{aligned}$$

Let us focus on the case $K = B_r(0)$. In the last section we sketch the computation of the Fourier transform of $\chi_{B_r(0)}$, resulting in

$$\widehat{\chi}_{B_r(0)}(\gamma) = \left(\frac{r}{\|\gamma\|}\right)^{\frac{d}{2}} J_{\frac{d}{2}}(2\pi r\|\gamma\|)\,,$$

where

$$J_\nu(z) = \frac{z^\nu}{2^\nu \pi^{\frac{1}{2}}\Gamma(\nu+\frac{1}{2})}\int_0^\pi \mathrm{e}^{-iz\cos(\alpha)}\sin(\alpha)^{2\nu}d\alpha\,, \quad \nu > -\frac{1}{2}$$

denotes the Bessel function of the first kind. The squared L^2-discrepancy of the standard lattice $\mathbb{Z}^d$ over balls can now be written as

$$\mathcal{D}_{B_r(0)}^{L^2}(\mathbb{Z}^d)^2 = r^d \sum_{\gamma\in\mathbb{Z}^d\setminus\{0\}} \frac{J_{\frac{d}{2}}(2\pi r\|\gamma\|)^2}{\|\gamma\|^d}\,.$$

Before moving on to the proofs of the main theorems, we will need asymptotics of the Bessel function $J_{d/2}$. For a proof of the following, see for instance Stein-Shakarchi's book "Complex Analysis", Appendix A, Section 1.

Lemma 4.1. *As $s \to +\infty$, one has*

$$J_\nu(s) = \sqrt{\frac{2}{\pi s}}\cos\left(s - \frac{2\nu+1}{4}\pi\right) + O(s^{-3/2})\,.$$

Since the series

$$\sum_{\gamma\in\mathbb{Z}^d\setminus\{0\}} \frac{1}{\|\gamma\|^s}$$

converges for every $s > d$, we get that

$$\mathcal{D}^{L^2}_{B_r(0)}(\mathbb{Z}^d)^2 = \frac{2}{\pi}\, r^{d-1} \sum_{\gamma\in\mathbb{Z}^d\setminus\{0\}} \frac{\cos^2(2\pi r\|\gamma\| - \frac{d+1}{4}\pi)}{\|\gamma\|^{d+1}} + O(r^{d-2})\,.$$

In particular, we can for every $\varepsilon > 0$ find an integer $N_\varepsilon \geq 1$ and a radius $r_\varepsilon > 0$ such that

$$\Big|\frac{\mathcal{D}^{L^2}_{B_r(0)}(\mathbb{Z}^d)^2}{r^{d-1}} - \frac{2}{\pi} \sum_{\substack{\gamma\in\mathbb{Z}^d\setminus\{0\}\\ \|\gamma\|\leq\sqrt{N_\varepsilon}}} \frac{\cos^2(2\pi r\|\gamma\| - \frac{d+1}{4}\pi)}{\|\gamma\|^{d+1}}\Big| < \varepsilon \tag{4.1}$$

for every $r \geq r_\varepsilon$.

5. Proof of Theorem 3.1 for $\mathbb{Z}^d$

We show that

$$\liminf_{R\to+\infty} \frac{1}{R}\int_0^R \frac{\mathcal{D}^{L^2}_{B_r(0)}(\mathbb{Z}^d)^2}{r^{d-1}}dr > 0\,,$$

from which it follows that there is a sequence $r_j \to +\infty$ such that

$$\inf_j \frac{\mathcal{D}^{L^2}_{B_{r_j}(0)}(\mathbb{Z}^d)}{r_j^{\frac{d-1}{2}}} > 0\,.$$

From Equation 4.1 and the fact that

$$\lim_{R\to+\infty} \frac{1}{R}\int_0^R \cos^2(2\pi r\|\gamma\| - \tfrac{d+1}{4}\pi)dr = \frac{1}{2}\,,$$

we get

$$\liminf_{R\to+\infty} \frac{1}{R}\int_0^R \frac{\mathcal{D}^{L^2}_{B_r(0)}(\mathbb{Z}^d)^2}{r^{d-1}}dr \geq \frac{1}{\pi} \sum_{\substack{\gamma\in\mathbb{Z}^d\setminus\{0\}\\ \|\gamma\|\leq\sqrt{N_\varepsilon}}} \frac{1}{\|\gamma\|^{d+1}} - \varepsilon\,,$$

which is positive for sufficiently small $\varepsilon > 0$.

6. Proof of Theorem 3.2

From Equation 4.1 we see that it suffices to prove that, if and only if $d \equiv 1 \bmod 4$, there is for every finite collection $\gamma_1, \ldots, \gamma_N \in \mathbb{Z}^d\setminus\{0\}$ a sequence $r_j \to +\infty$ such that

$$\lim_{j\to+\infty} \cos(2\pi r_j\|\gamma_n\| - \tfrac{d+1}{4}\pi) = 0$$

for every $n = 1, \ldots, N$.

First assume that $d \equiv 1 \bmod 4$. Then $(d+1)/4$ is an odd multiple of $\pi/2$, so

$$\cos(2\pi r\|\gamma\| - \tfrac{d+1}{4}\pi) = \pm\sin(2\pi r\|\gamma\|)\,.$$

Consider the continuous surjective map $S : \mathbb{R}^N \to [0,1]^N$ given by

$$S(x_1, \ldots, x_N) = (\sin(2\pi x_1), \ldots, \sin(2\pi x_N))\,.$$

This map is $\mathbb{Z}^N$-periodic, so it suffices to find a sequence $r_j \to +\infty$ such that

$$(r_j\gamma_1, \ldots, r_j\gamma_N) + \mathbb{Z}^N \to (0, \ldots, 0) + \mathbb{Z}^N$$

in the torus $\mathbb{R}^N/\mathbb{Z}^N$ as $j \to +\infty$. By compactness there is a sequence $r'_k \to +\infty$ such that $(r'_k\gamma_1, \ldots, r'_k\gamma_N) + \mathbb{Z}^N$ converges as $k \to +\infty$, and taking subsequences $k_j, \ell_j \to +\infty$ such that

$$r_j := r'_{k_j} - r'_{\ell_j} \geq j$$

we get $r_j \to +\infty$ and

$$\lim_{j\to+\infty} (r_j\gamma_1, \ldots, r_j\gamma_N) + \mathbb{Z}^N = (0, \ldots, 0) + \mathbb{Z}^N$$

as desired.

For the converse implication, suppose $d \not\equiv 1 \bmod 4$ and assume that there is a sequence $r_j \to +\infty$ such that

$$\lim_{j\to+\infty} \cos(2\pi r_j\|\gamma\| - \tfrac{d+1}{4}\pi) = 0$$

for some $\gamma \in \mathbb{Z}^d\backslash\{0\}$. Without loss of generality, we can assume that $r_j\|\gamma\| + \mathbb{Z} \to \frac{d+3}{8} + \mathbb{Z}$ in the unit circle $\mathbb{R}/\mathbb{Z}$. Then for $2\gamma \in \mathbb{Z}^d\backslash\{0\}$ we get

$$\lim_{j\to+\infty} \cos(2\pi r_j\|2\gamma\| - \tfrac{d+1}{4}\pi) = \cos(\tfrac{d+5}{4}\pi) \neq 0$$

and we're done.

7. The bigger picture: Dynamical irregularites of distribution

The key ingredients we used for understanding the L^2-discrepancy of the standard lattice were

(1) the (compact) space $\mathbb{R}^d/\mathbb{Z}^d$, measurably equivalent to $[0,1)^d$, with its translation invariant probability measure, and

(2) the spectral formula

$$\mathcal{D}^{L^2}_{B_r(0)}(\mathbb{Z}^d)^2 = \sum_{\gamma\in\mathbb{Z}^d\backslash\{0\}} |\widehat{\chi}_{B_r(0)}(\gamma)|^2\,.$$

These have analogues for a more general class of locally finite point sets $P \subset \mathbb{R}^d$ as we describe now.

Let $\mathrm{LF}(\mathbb{R}^d)$ denote the space of locally finite subsets of $\mathbb{R}^d$ endowed with the *vague topology*, meaning that a sequence $P_n \in \mathrm{LF}(\mathbb{R}^d)$ converges to $P \in \mathrm{LF}(\mathbb{R}^d)$ if for every continuous compactly supported $f \in C_c(\mathbb{R}^d)$, one has

$$\sum_{p \in P_n} f(p) \longrightarrow \sum_{p \in P} f(p) \,.$$

It is then easy to check that the action of $\mathbb{R}^d$ on $\mathrm{LF}(\mathbb{R}^d)$ by translations is a continuous group action.

Fact: The space $M_+(\mathbb{R}^d; \mathbb{Z})$ of integer-valued positive measures on $\mathbb{R}^d$ with the vague topology is a Polish space (i.e. complete, separable and metrizable), see for instance [4, Lemma 9.1.V]. However, we keep in mind that $\mathrm{LF}(\mathbb{R}^d) \hookrightarrow M_+(\mathbb{R}^d; \mathbb{Z})$ is not complete in the vague topology, a simple counterexample being $\delta_{1/n} + \delta_0 \to 2\delta_0$.

Given $P \in \mathrm{LF}(\mathbb{R}^d)$, one constructs its *hull*

$$\Omega_P = \overline{\{t + P \in \mathrm{LF}(\mathbb{R}^d) \,|\, t \in \mathbb{R}^d\}} \,,$$

where the closure is taken in the vague topology, and one verifies that the action of $\mathbb{R}^d$ preserves Ω_P. Note that one might have $\varnothing \in \Omega_P$ if P has arbitrarily large "gaps" (more precisely when P is not *relatively dense*). Moreover, if P is *uniformly discrete*, which means that "gaps" have sizes uniformly bounded from below, one can show that Ω_P is compact. For the standard lattice $\mathbb{Z}^d$, we recover the torus $\Omega_{\mathbb{Z}^d} = \mathbb{R}^d/\mathbb{Z}^d$. In particular, $\varnothing \notin \Omega_{\mathbb{Z}^d}$. In general, the hull Ω_P does not enjoy any specific properties, but if one restricts their attention to the class of *Delone sets*, meaning uniformly discrete and relatively dense subsets, then the hull is compact and does not contain the empty set. Compactness ensures the existence of invariant probability measures on Ω_P, which is what we will use to define the L^2-discrepancy in this context.

Lemma 7.1 (Amenability). *If $P \in \mathrm{LF}(\mathbb{R}^d)$ such that Ω_P is compact, then there is a translation invariant probability measure on Ω_P.*

If $\varnothing \in \Omega_P$ then $\delta_\varnothing$ is an invariant measure on Ω_P, so we say that μ is *non-trivial* if $\mu(\{\varnothing\}) = 0$ in order to avoid this degenerate case. Moreover, such a non-trivial measure μ is *ergodic* if $\mu(E) \in \{0, 1\}$ for every translation invariant Borel subset $E \subset \Omega_P$. For example, the invariant probability measure on $\Omega_{\mathbb{Z}^d} = \mathbb{R}^d/\mathbb{Z}^d$ is non-trivial and ergodic.

Fact:(Extremality) There is an ergodic translation invariant probability measure μ on Ω_P.

Given that Ω_P admits a non-trivial invariant probability μ, there is first of all a constant $i_P > 0$ such that

$$\int_{\Omega_P} \#(P' \cap K) d\mu(P') = i_P |K|$$

for every compact $K \subset \mathbb{R}^d$ with non-empty interior. The constant i_P is in a sense the natural generalization of the *covolume* of a lattice. Secondly, we then define the L^2-discrepancy of (Ω_P, μ) to be

$$\mathcal{D}_K^{L^2}(\mu) = \Big(\int_{\Omega_P} \Big| \#(P' \cap K) - i_P|K| \Big|^2 d\mu(P') \Big)^{1/2} ,$$

given that this quantity is well-defined. The existence of a spectral formula for this general situation can be answered through the Bochner-Schwartz Theorem.

Theorem 7.2 (Bochner-Schwartz). *Let (Ω_P, μ) be a non-trivial measured hull. Then there is a unique non-trivial positive Radon measure σ_μ on $\mathbb{R}^d$ such that*

$$\int_{\Omega_P} \Big| \sum_{p \in P'} f(p) - i_P \int_{\mathbb{R}^d} f(t) dt \Big|^2 d\mu(P') = \int_{\mathbb{R}^d} |\widehat{f}(\omega)|^2 d\sigma_\mu(\omega)$$

for every Schwartz function $f \in \mathscr{S}(\mathbb{R}^d)$. Moreover, if μ is ergodic then $\sigma_\mu(\{0\}) = 0$.

In particular, the L^2-discrepancy satisfies

$$\mathcal{D}_K^{L^2}(\mu)^2 = \int_{\mathbb{R}^d} |\widehat{\chi}_K(\omega)|^2 d\sigma_\mu(\omega) .$$

The definition of the L^2-discrepancy and this spectral formula extends more generally to any translation invariant probability measures μ on $\mathrm{LF}(\mathbb{R}^d)$, in other words to *invariant point processes* on $\mathbb{R}^d$. For the standard lattice we had

$$\sigma_{\mathbb{Z}^d} = \sum_{\gamma \in \mathbb{Z}^d \setminus \{0\}} \delta_\gamma .$$

Beck's Theorem, Theorem 3.1, now extends to non-trivial measured hulls and moreover to the setting of general invariant point processes on $\mathbb{R}^d$.

Remark 7.3. The largest generality for which all of this machinery works is for *invariant locally square-integrable random measures* on $\mathbb{R}^d$. The space $\mathscr{M}_+(\mathbb{R}^d)$ of positive Radon measures on $\mathbb{R}^d$ is Polish in the vague topology and a translation invariant probability measure μ on $\mathscr{M}_+(\mathbb{R}^d)$ is *locally square-integrable* if

$$\int p(K)^2 d\mu(p) < +\infty$$

for all compact $K \subset \mathbb{R}^d$. There is a constant $i_\mu > 0$, usually called the *intensity* of μ, such that

$$\int p(K) d\mu(p) = i_\mu |K| \,,$$

and the L^2-discrepancy

$$\mathcal{D}_K^{L^2}(\mu) = \Big(\int \Big| p(K) - i_\mu |K| \Big|^2 d\mu(p) \Big)^{1/2}$$

is well-defined, admitting a unique positive Radon measure σ_μ, usually called the *Bartlett spectrum*, satisfying

$$\mathcal{D}_K^{L^2}(\mu)^2 = \int_{\mathbb{R}^d} |\widehat{\chi}_K(\omega)|^2 d\sigma_\mu(\omega) \,.$$

for every compact K.

Theorem 7.4 (Beck, Björklund-B.)**.** *Let μ be a translation invariant (ergodic) locally square-integrable random measure on $\mathbb{R}^d$. Then*

$$\limsup_{r \to +\infty} \frac{\mathcal{D}_{B_r(0)}^{L^2}(\mu)}{r^{\frac{d-1}{2}}} > 0 \,.$$

Proof. We show as in the lattice case that

$$\liminf_{R \to +\infty} \frac{1}{R} \int_0^R \frac{\mathcal{D}_{B_r(0)}^{L^2}(\mu)^2}{r^{d-1}} dr > 0 \,.$$

To see this, first use the spectral formula,

$$\frac{1}{R} \int_0^R \frac{\mathcal{D}_{B_r(0)}^{L^2}(\mu)^2}{r^{d-1}} dr = \int_{\mathbb{R}^d} \Big(\frac{1}{R} \int_0^R J_{\frac{d}{2}}(2\pi r \|\omega\|)^2 r dr \Big) \frac{d\sigma_\mu(\omega)}{\|\omega\|^{d+1}} \,.$$

By Fatou's lemma it suffices to show that

$$\liminf_{R \to +\infty} \frac{1}{R} \int_0^R J_{\frac{d}{2}}(2\pi r \|\omega\|)^2 r dr > 0$$

for every $\omega \in \mathbb{R}^d$. This integral was first computed by Lommel as

$$\frac{1}{R} \int_0^R J_{\frac{d}{2}}(2\pi r \|\omega\|)^2 r dr = \frac{R}{2} \Big(J_{\frac{d}{2}}(2\pi R \|\omega\|)^2 - J_{\frac{d-2}{2}}(2\pi R \|\omega\|) J_{\frac{d+2}{2}}(2\pi R \|\omega\|) \Big)$$

and using the asymptotic expansion in Lemma 4.1 we get that

$$\begin{aligned} &\frac{R}{2} \Big(J_{\frac{d}{2}}(2\pi R \|\omega\|)^2 - J_{\frac{d}{2}}(2\pi R \|\omega\|) J_{\frac{d}{2}}(2\pi R \|\omega\|) \Big) = \\ &= \frac{1}{\pi} \Big(\cos^2(2\pi R \|\omega\| - \tfrac{d+1}{4}\pi)) - \cos(2\pi R \|\omega\| - \tfrac{d-1}{4}\pi)) \cos(2\pi R \|\omega\| - \tfrac{d+3}{4}\pi)) \Big) + O_{d,\omega}(R^{-1}) \end{aligned}$$

$$= \frac{1}{\pi} + O_{d,\omega}(R^{-1})\,.$$

Finally, we have shown that

$$\liminf_{R\to+\infty} \frac{1}{R}\int_0^R \frac{\mathcal{D}^{L^2}_{B_r(0)}(\mu)^2}{r^{d-1}}dr \geq \frac{1}{\pi}\int_{\mathbb{R}^d} \frac{d\sigma_\mu(\omega)}{\|\omega\|^{d+1}} > 0\,.$$

□

Remark 7.5. The integral

$$\int_{\mathbb{R}^d} \frac{d\sigma_\mu(\omega)}{\|\omega\|^{d+1}}$$

can be infinite, and the potential divergence is a result of the behaviour of σ_μ close to 0. If the integral is finite then μ is *hyperuniform* in the sense that

$$\limsup_{r\to+\infty} \frac{\mathcal{D}^{L^2}_{B_r(0)}(\mu)}{r^{\frac{d}{2}}} = 0\,.$$

The notion of hyperuniformity has relevance in modern statistical physics and its applications are investigated actively to this day.

8. Sketch: Computing the Fourier transform of the indicator of a ball

The Fourier transform of the indicator $\chi_{B_r(0)}$ can be computed by introducing polar coordinates,

$$\widehat{\chi}_{B_r(0)}(\gamma) = \int_0^r \Big(\int_{\mathbb{S}^{d-1}} \mathrm{e}^{-2\pi it\langle u,\gamma\rangle} d\mathcal{H}^{d-1}(u) \Big) t^{d-1} dt\,.$$

Here $\mathcal{H}^{d-1}$ denotes the Hausdorff/surface measure on $\mathbb{S}^{d-1}$. The inner integral can be computed using standard coordinates on the unit sphere as

$$\int_{\mathbb{S}^{d-1}} \mathrm{e}^{-2\pi it\langle u,\gamma\rangle} d\mathcal{H}^{d-1}(u) = \mathcal{H}^{d-2}(\mathbb{S}^{d-2}) \int_0^\pi \mathrm{e}^{-2\pi it\|\gamma\|\cos(\alpha)} \sin(\alpha)^{d-2} d\alpha\,,$$

and if we consider Bessel functions of the first kind,

$$J_\nu(z) = \frac{z^\nu}{2^\nu \pi^{\frac{1}{2}} \Gamma(\nu+\frac{1}{2})} \int_0^\pi \mathrm{e}^{-iz\cos(\alpha)} \sin(\alpha)^{2\nu} d\alpha$$

then

$$\begin{aligned}\widehat{\chi}_{B_r(0)}(\gamma) &= \frac{2^{\frac{d-2}{2}} \pi^{\frac{1}{2}} \Gamma(\frac{d-1}{2})}{(2\pi\|\gamma\|)^{\frac{d-2}{2}}} \mathcal{H}^{d-2}(\mathbb{S}^{d-2}) \int_0^r J_{\frac{d-2}{2}}(2\pi s\|\gamma\|) s^{\frac{d}{2}} ds \\ &= \frac{2^{\frac{d-2}{2}} \pi^{\frac{1}{2}} \Gamma(\frac{d-1}{2})}{(2\pi\|\gamma\|)^{d}} \mathcal{H}^{d-2}(\mathbb{S}^{d-2}) \int_0^{2\pi r\|\gamma\|} J_{\frac{d-2}{2}}(s) s^{\frac{d}{2}} ds\,.\end{aligned}$$

Having a look at your favorite table of integrals of Bessel functions, one finds

$$\int_0^r J_\nu(s) s^{\nu+1} ds = J_{\nu+1}(r) r^{\nu+1}$$

and keeping track of all constants, the formula for $\widehat{\chi}_{B_r(0)}$ simplifies to

$$\widehat{\chi}_{B_r(0)}(\gamma) = \Big(\frac{r}{\|\gamma\|}\Big)^{\frac{d}{2}} J_{\frac{d}{2}}(2\pi r\|\gamma\|)\,.$$

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