

L^2 -DISCREPANCIES IN THE HYPERBOLIC PLANE

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Structure of the talk:

- (1) Recap on point processes and Beck's Theorem
- (2) The hyperbolic plane $\mathbb{H}^2$
- (3) Zonal spherical functions on $\mathbb{H}^2$
- (4) Spectral measures
- (5) Asymptotics for the spherical transform of a ball
- (6) A hyperbolic analogue of Beck's Theorem

1. INVARIANT POINT PROCESSES

Let (X, d) be a proper metric space (i.e. closed balls are compact) and $\mathcal{M}_+(\mathbb{R}^d)$ the space of positive Radon measures on X endowed with the vague topology, meaning that convergence $p_n \rightarrow p$ of a sequence of Radon measures is given by

$$\int_X f(x) dp_n(x) \longrightarrow \int_X f(x) dp(x)$$

for every $f \in C_c(X)$ as $n \rightarrow +\infty$. Then $\mathcal{M}_+(X)$ is Polish and the subspace $\mathcal{M}_+(X, \mathbb{Z}_{\geq 0}) \subset \mathcal{M}_+(X)$ of integer-valued measures is closed and hence Polish in the subspace topology [6, Lemma 9.1.V].

Definition 1.1. A *random measure* on X is a probability measure μ on $\mathcal{M}_+(X)$, and such a measure is a *point process* if $\text{supp}(\mu) \subset \mathcal{M}_+(X, \mathbb{Z}_{\geq 0})$. Moreover, a point process μ is *simple* if it assigns measure 1 to the measurable subset $\text{LF}(X)$ of (Dirac measures over) locally finite subsets of X .

We will be interested in the situation when G is a group of isometries acting transitively on X , in which case we consider G -invariant random measures μ on X . In the Euclidean case we have some concrete examples:

Example 1.1 (Dynamical hulls). Hulls $\Omega_P = \overline{\mathbb{R}^d + P}$ of point sets $P \in \text{LF}(\mathbb{R}^d)$ that admit an invariant measure μ . In special situations, such as $P = \mathbb{Z}^d$, such a measure μ is unique (and necessarily ergodic). Uniqueness is also guaranteed for *model sets* in the sense of Meyer, also called *cut-and-project sets* or *quasicrystals*, constructed from an irreducible lattice $\Gamma < \mathbb{R}^d \times \mathbb{R}^{d'}$ and a compact subset $W \subset \mathbb{R}^{d'}$ with non-empty interior as

$$P = \text{Proj}_{\mathbb{R}^d}(\Gamma \cap (\mathbb{R}^d \times W)).$$

The hull Ω_P is identified with the torus $(\mathbb{R}^d \times \mathbb{R}^{d'})/\Gamma$ and the unique $\mathbb{R}^d$ -invariant measure is induced by the Haar measure. A basic example in dimension 1 is the subset $P \subset \mathbb{Z}[\sqrt{2}]$ of *Pisot-Salem numbers* for the number field $\mathbb{Q}(\sqrt{2})$, i.e.

$$P = \left\{ m + n\sqrt{2} \in \mathbb{Z}[\sqrt{2}] \mid |m - n\sqrt{2}| \leq 1 \right\},$$

for which

$$\Gamma = \begin{pmatrix} 1 & \sqrt{2} \\ 1 & -\sqrt{2} \end{pmatrix} \mathbb{Z}^2, \quad W = [-1, 1].$$

In ongoing work with M. Björklund we prove uniqueness of an invariant measure μ for hulls of *Fourier quasicrystals* P in $\mathbb{R}^d$, which are point sets satisfying a generalization of the Poisson summation formula,

$$\sum_{p \in P} f(p) = \sum_{s \in S} c(s) \hat{f}(s), \quad f \in \mathcal{S}(\mathbb{R}^d), \quad c : S \rightarrow \mathbb{C}$$

where $S \subset \mathbb{R}^d$ is locally finite. The first example of such sets P were constructed in 2020 by Kurasov-Sarnak [5].

We also give an example which is more probabilistic in nature.

Example 1.2 (Poisson point processes). Given $m \in \mathcal{M}_+(X)$ one can construct a probability measure μ_m on $\mathcal{M}_+(X)$, called the *Poisson point process* with *intensity measure* m . It is uniquely defined as the simple point process for which counting functions $N_K : P \mapsto \#(P \cap K)$ over compact K are Poisson distributed with parameter $m(K)$ and such that $N_K, N_{K'}$ are independent whenever $K \cap K' = \emptyset$.

When $X = \mathbb{R}^d$ and $dm(t) = \lambda dt$ is a positive multiple of the Lebesgue measure, the associated Poisson point process μ_m is an invariant simple point process on $\mathbb{R}^d$.

With some examples in mind we let μ be an invariant general random measure (or a simple point process) on $\mathbb{R}^d$ and assume that it is *locally square-integrable* in the sense that

$$\int p(K)^2 d\mu(p) < +\infty$$

for all compact $K \subset \mathbb{R}^d$. Then the L^2 -discrepancy

$$\mathcal{D}_K^{L^2}(\mu) = \left(\int \left| p(K) - \rho_\mu |K| \right|^2 d\mu(p) \right)^{1/2}$$

is well-defined, where

$$\rho_\mu = \int p([0, 1]^d) d\mu(p)$$

is the average density of μ . As mentioned in the last talk, the Bochner-Schwartz Theorem applied to the translation invariant positive-definite bilinear form

$$\beta_\mu(f \otimes \bar{f}) = \int \left| \int_{\mathbb{R}^d} f(t) dp(t) - \rho_\mu \int_{\mathbb{R}^d} f(t) dt \right|^2 d\mu(p)$$

ensures the existence of a spectral measure σ_μ on $\mathbb{R}^d$ such that

$$\beta_\mu(f \otimes \bar{f}) = \int_{\mathbb{R}^d} |\widehat{f}(\omega)|^2 d\sigma_\mu(\omega).$$

In particular, the L^2 -discrepancy over a compact set K satisfies

$$\mathcal{D}_K^{L^2}(\mu)^2 = \int_{\mathbb{R}^d} |\widehat{\chi}_K(\omega)|^2 d\sigma_\mu(\omega).$$

Example 1.3. For the examples of point processes mentioned earlier, the corresponding spectral measure is the following:

- (1) For the hull of the standard lattice $\mathbb{Z}^d$,

$$\sigma_{\mathbb{Z}^d} = \sum_{\gamma \in \mathbb{Z}^d \setminus \{0\}} \delta_\gamma.$$

- (2) For the hull of a model set $P = \text{Proj}_{\mathbb{R}^d}(\Gamma \cap (\mathbb{R}^d \times W))$ one has

$$\sigma_P = \sum_{(\omega_1, \omega_2) \in \Gamma^\perp \setminus \{(0,0)\}} |\widehat{\chi}_W(\omega_2)|^2 \delta_{\omega_1}, \quad (\text{dense set of atoms!}).$$

- (3) For the invariant measure on the hull of a Fourier quasicrystal P with spectrum S ,

$$\sigma_P = \sum_{s \in S} |c(s)|^2 \delta_s.$$

- (4) For the invariant Poisson point process μ with intensity 1,

$$d\sigma_\mu(\omega) = d\omega.$$

Beck's Theorem from the last talk can be stated in this setting of random measures and point processes [1, Theorem 2A].

Theorem 1.2 (Beck, Björklund-B.). *Let μ be a translation invariant (ergodic) locally square-integrable random measure on $\mathbb{R}^d$. Then*

$$\limsup_{r \rightarrow +\infty} \frac{\mathcal{D}_{B_r(0)}^{L^2}(\mu)}{r^{\frac{d-1}{2}}} > 0.$$

For a complete proof of this, see the notes for the previous talk. Here, we give a brief sketch.

Sketch of Proof. We show as in the lattice case that

$$\liminf_{R \rightarrow +\infty} \frac{1}{R} \int_0^R \frac{\mathcal{D}_{B_r(0)}^{L^2}(\mu)^2}{r^{d-1}} dr > 0.$$

To see this, first use the spectral formula,

$$\frac{1}{R} \int_0^R \frac{\mathcal{D}_{B_r(0)}^{L^2}(\mu)^2}{r^{d-1}} dr = \int_{\mathbb{R}^d} \left(\frac{1}{R} \int_0^R |\widehat{\chi}_{B_r(0)}(\omega)|^2 dr \right) d\sigma_\mu(\omega).$$

One uses the asymptotic expansion of Bessel functions of the first kind to compute

$$\lim_{R \rightarrow +\infty} \frac{1}{R} \int_0^R |\widehat{\chi}_{B_r(0)}(\omega)|^2 dr = \frac{1}{\pi \|\omega\|^{d+1}}$$

for every $\omega \in \mathbb{R}^d$, so by Fatou's Lemma we get

$$\liminf_{R \rightarrow +\infty} \frac{1}{R} \int_0^R \frac{\mathcal{D}_{B_r(0)}^{L^2}(\mu)^2}{r^{d-1}} dr \geq \frac{1}{\pi} \int_{\mathbb{R}^d} \frac{d\sigma_\mu(\omega)}{\|\omega\|^{d+1}} > 0.$$

□

To summarize, the main ingredients for proving a Beck-type Theorem for invariant point processes are

- (1) A spectral characterization of the L^2 -discrepancy,

$$\mathcal{D}_{B_r(0)}^{L^2}(\mu)^2 = \int_{\mathbb{R}^d} |\widehat{\chi}_{B_r(0)}(\omega)|^2 d\sigma_\mu(\omega).$$

- (2) The averaging property

$$\liminf_{R \rightarrow +\infty} \frac{1}{R} \int_0^R |\widehat{\chi}_{B_r(0)}(\omega)|^2 dr > 0$$

for every $\omega \in \mathbb{R}^d$.

The aim is now to establish these two properties for invariant point processes in the hyperbolic plane. As we will see, some new behavior emerges.

2. THE HYPERBOLIC PLANE

The hyperbolic plane, denoted $\mathbb{H}^2$, is the metric universal cover of surfaces with fixed constant negative curvature, much like the Euclidean plane is the metric universal cover of flat tori. The hyperbolic plane has several models in which it is

realized explicitly, and we will mainly be interested in the *upper-half plane model* and *disk model* of $\mathbb{H}^2$.

2.1. The upper-half plane model

Consider the upper-half plane as a subset of the complex numbers,

$$\mathbb{C}^+ = \{z \in \mathbb{C} \mid \text{Im}(z) > 0\}$$

with the action of the group $G = \text{SL}_2(\mathbb{R})$ by Möbius transformations,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az + b}{cz + d}, \quad ad - bc = 1.$$

This action is orientation-preserving and isometric with respect to the metric

$$d(z_1, z_2) = 2 \operatorname{arcsinh} \left(\frac{|z_1 - z_2|}{2\sqrt{\text{Im}(z_1)\text{Im}(z_2)}} \right). \quad (\text{check that it is indeed a metric!}).$$

The stabilizer of the reference point $i \in \mathbb{C}^+$ is the compact subgroup

$$K = \text{Stab}_G(i) = \left\{ k_v = \begin{pmatrix} \cos(v) & -\sin(v) \\ \sin(v) & \cos(v) \end{pmatrix} \in G \mid v \in [0, 2\pi) \right\} = \text{SO}(2).$$

We can introduce natural coordinates for $\mathbb{H}^2 = (\mathbb{C}^+, d)$ in the following way: By QR-factorization, we have the *Iwasawa decomposition*

$$G = \text{SL}_2(\mathbb{R}) = NAK$$

where

$$A = \left\{ a_t = \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix} \in G \mid t \in \mathbb{R} \right\}$$

and

$$N = \left\{ n_s = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \in G \mid s \in \mathbb{R} \right\}.$$

In particular, $z = n_{\text{Re}(z)} a_{\log(\text{Im}(z))} \cdot i$ for every $z \in \mathbb{C}^+$ and so the action of $AN < G$ on $\mathbb{C}^+$ is transitive. Geometrically, the curve $\alpha(t) = a_t \cdot i = e^t \cdot i$ is a bi-infinite geodesic from 0 to the ideal point ∞ at infinity and $\theta(s) = n_s \cdot i = i + s$ is a *horocycle* tangent to ∞ , meaning a curve that is perpendicular to every bi-infinite geodesic emanating from the ideal point ∞ . Every bi-infinite geodesic in $(\mathbb{C}^+, d)$ can be realized as $t \mapsto g a_t \cdot i$ for some $g \in G$ and every horocycle tangent to some point in the ideal boundary $\partial\mathbb{H}^2 = \mathbb{R} \cup \{\infty\}$ can be realized as $s \mapsto g n_s \cdot i$ for some $g \in G$.

In order to do harmonic analysis over $\mathbb{H}^2$ we'll need an explicit Haar measure on G , or equivalently as we will see, an invariant measure on $\mathbb{H}^2$. To do this, let m be the Lebesgue measure on $\mathbb{C}$. The derivative of a Möbius transformation $g \in \text{SL}_2(\mathbb{R})$ is

$$\partial(g.z) = \frac{1}{(cz + d)^2} = \left(\frac{|cz + d|}{(cz + d)} \right)^2 \text{Im}(g.z), \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

so the measure $dm_{\mathbb{C}^+}(z) = \text{Im}(z)^{-2}dm(z)$ is $\text{SL}_2(\mathbb{R})$ -invariant as

$$\begin{aligned} \int_{\mathbb{H}^2} f(g.z) \frac{dm(z)}{\text{Im}(z)^2} &= \int_{\mathbb{H}^2} f(g.z) \frac{\text{Im}(g.z)^2}{\text{Im}(z)^2} \frac{dm(z)}{\text{Im}(g.z)^2} \\ &= \int_{\mathbb{H}^2} f(g.z) \left| \frac{\partial(g.z)}{\partial z} \right|^2 \frac{dm(z)}{\text{Im}(g.z)^2} = \int_{\mathbb{H}^2} f(z) \frac{dm(z)}{\text{Im}(z)^2} \end{aligned}$$

for every $f \in C_c(\mathbb{C}^+)$. Thus we get a Haar measure m_G on $G = \text{SL}_2(\mathbb{R})$ by the formula

$$\int_G \varphi(g) dm_G(g) = \int_0^{2\pi} \int_{\mathbb{H}^2} \varphi(n_{\text{Re}(z)} a_{\log(\text{Im}(z))} k_v) \frac{dm(z) dv}{\text{Im}(z)^2}.$$

Setting $s = \text{Re}(z)$, $t = \log(\text{Im}(z))$ yields a formula for m_G in Iwasawa coordinates,

$$dm_G(n_s a_t k_v) = e^{-t} ds dt dv \quad s, t \in \mathbb{R}, v \in [0, 2\pi).$$

2.2. The unit disk model

Consider now the unit disk

$$\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$$

with the action of the group $G = \text{SU}(1, 1)$ by Möbius transformations,

$$\begin{pmatrix} \alpha & \bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix} . z = \frac{\alpha z + \bar{\beta}}{\beta z + \bar{\alpha}}, \quad |\alpha|^2 - |\beta|^2 = 1.$$

This action is orientation-preserving and isometric with respect to the metric

$$d(z_1, z_2) = 2 \operatorname{arcsinh} \left(\frac{|z_1 - z_2|}{\sqrt{(1 - |z_1|^2)(1 - |z_2|^2)}} \right).$$

The relation between the disk model $(\mathbb{D}, d)$ and the upper-half plane model $(\mathbb{C}^+, d)$ of $\mathbb{H}^2$ is made explicit by a conformal equivalence known as the *Cayley transform* $C : \mathbb{C}^+ \rightarrow \mathbb{D}$ given by

$$C(z) = \frac{z - i}{z + i}.$$

Moreover, it identifies the ideal boundary $\mathbb{R} \cup \{\infty\}$ of $\mathbb{C}^+$ with the boundary $\partial\mathbb{D} = \mathbb{S}^1$ of $\mathbb{D}$.

The corresponding stabilizer $K < G$ of the reference point $0 \in \mathbb{D}$ is

$$K = \text{Stab}_G(0) = \left\{ k_v = \begin{pmatrix} e^{iv/2} & 0 \\ 0 & e^{-iv/2} \end{pmatrix} \in G \mid v \in [0, 2\pi) \right\} = \text{U}(1)$$

and the corresponding subgroups A, N in the Iwasawa decomposition $G = NAK$ are

$$A = \left\{ \begin{pmatrix} \cosh(t/2) & \sinh(t/2) \\ \sinh(t/2) & \cosh(t/2) \end{pmatrix} \in G \mid t \in \mathbb{R} \right\}$$

and

$$N = \left\{ \begin{pmatrix} 1 + \frac{is}{2} & -\frac{is}{2} \\ \frac{is}{2} & 1 - \frac{is}{2} \end{pmatrix} \in G \mid s \in \mathbb{R} \right\}.$$

Note that the action of K , $k_v.z = e^{iv}z$ is very natural in this model, and that the actions of A, N are more well-suited for the upper-half plane model. Just to note it, the reference bi-infinite geodesic here is $\alpha(t) = a_t.0 = \tanh(t/2)$ and the reference horocycle tangent to $C(\infty) = 1$ is $\theta(s) = n_s.0 = s/(s + 2i)$.

There is however a more natural coordinate system in the unit disk model: By singular value factorization we have the *Cartan decomposition*

$$G = \mathrm{SU}(1, 1) = KA^+K$$

where $A^+ = \{a_t \in A \mid t \geq 0\}$. In particular, $z = k_{\arg(z)}a_{d(z,0)}.0$ for every $z \in \mathbb{D}$, so the action of $KA < G$ on $\mathbb{D}$ is transitive. From this we get an invariant measure

$$dm_{\mathbb{D}}(z) = \frac{4 dm(z)}{(1 - |z|^2)^2}$$

by the same argument as for the upper-half plane model, and a Haar measure for $G = \mathrm{SU}(1, 1)$ in Cartan coordinates is

$$\int_G \varphi(g) dm_G(g) = 4 \int_0^{2\pi} \int_{\mathbb{D}} \varphi(k_{\arg(z)}a_{d(z,0)}k_v) \frac{dm(z)dv}{(1 - |z|^2)^2}.$$

We simplify this in two steps: first, introduce polar coordinates, so that

$$\int_{\mathbb{D}} \varphi(k_{\arg(z)}a_{d(z,0)}k_v) \frac{dm(z)}{(1 - |z|^2)^2} = \int_0^{2\pi} \int_0^1 \varphi(k_{v'}a_{t(r)}k_v) \frac{r dr dv}{(1 - r^2)^2}$$

where $t(r) = 2 \operatorname{arcsinh}(r/(1 - r^2)^{1/2})$ and $r = \tanh(t(r)/2)$. Making the substitution $t = t(r)$ yields

$$\begin{aligned} \int_0^{2\pi} \int_0^1 \varphi(k_{v'}a_{t(r)}k_v) \frac{r dr dv}{(1 - r^2)^2} &= \int_0^{2\pi} \int_0^\infty \varphi(k_{v'}a_t k_v) \frac{\tanh(t/2) dv' dt dv}{\cosh^2(t/2)(1 - \tanh^2(t/2))^2} \\ &= \frac{1}{2} \int_0^{2\pi} \int_0^\infty \varphi(k_{v'}a_t k_v) \sinh(t) dv' dt dv. \end{aligned}$$

In total, we have that the Haar measure m_G is given in Cartan coordinates by

$$dm_G(k_{v'}a_t k_v) = 2 \sinh(t) dv' dt dv.$$

In particular, the area of balls in $\mathbb{H}^2$ is

$$|B_r(0)| = 8\pi^2 \int_0^r \sinh(t) dt = 8\pi^2 \cosh(r) \sim 4\pi^2 e^r.$$

Moreover, the length of (centered) circles are

$$|\partial B_r(0)| = 2 \sinh(r) \sim e^r.$$

Thus Gauss' estimate for counting points in balls do not yield any interesting result. It is not even immediately clear that there are infinite relatively dense point sets $P \subset \mathbb{H}^2$ which admit a density.

3. ZONAL SPHERICAL FUNCTIONS ON $\mathbb{H}^2$

From now on we consider both models simultaneously, unless told otherwise, and use the notation $(\mathbb{H}^2, o, d, m_{\mathbb{H}^2})$ to mean either $(\mathbb{C}^+, i, d, m_{\mathbb{C}^+})$ or $(\mathbb{D}, 0, d, m_{\mathbb{D}})$.

3.1. The Fourier-Helgason-Laplace transform

The notion of Fourier transform on $\mathbb{H}^2$ that we will make use of is governed by the *Poisson kernel*

$$\mathcal{P}(z, \xi) = \begin{cases} \frac{\operatorname{Im}(z)}{|z - \xi|^2}, & \text{in } \mathbb{C}^+ \\ \frac{1 - |z|^2}{|z - \xi|^2}, & \text{in } \mathbb{D}, \end{cases}$$

where $\xi \in \partial\mathbb{H}^2$ is an element of the respective (ideal) boundary. One can think of $\mathcal{P}(g.o, \xi)$ as the Jacobian for the map g on the boundary $\partial\mathbb{H}^2$ evaluated at ξ . The crucial property here is that $z \mapsto \mathcal{P}(z, \xi)$ is harmonic for the relevant Laplacian (check!)

$$\Delta = \begin{cases} 4\operatorname{Im}(z)^2 \partial_z \bar{\partial}_z, & \text{in } \mathbb{C}^+ \\ (1 - |z|^2)^2 \partial_z \bar{\partial}_z, & \text{in } \mathbb{D}. \end{cases}$$

Moreover, if $s \in \mathbb{C}$ then

$$-\Delta \mathcal{P}(\cdot, \xi)^s = s(1 - s) \mathcal{P}(\cdot, \xi)^s \quad (\text{check!})$$

for every $\xi \in \partial\mathbb{H}^2$. Without further motivation, we introduce the *Fourier-Helgason-Laplace transform* $\widehat{\cdot}: C_c(G) \rightarrow C_0(\mathbb{C} \times \partial\mathbb{H}^2)$ by

$$\widehat{f}(s, \xi) = \int_{\mathbb{H}^2} f(z) \mathcal{P}(z, \xi)^s dm_{\mathbb{H}^2}(z)$$

where $m_{\mathbb{H}^2}$ is the invariant measure on $\mathbb{H}^2$ in the respective models. This transform decomposes a function f into its different "frequency parameters" (s, ξ) , but does

unfortunately not satisfy the useful "convolution-to-product" property that the Euclidean Fourier-Laplace transform does, and neither the "involution-to-conjugation" property that the Euclidean Fourier transform does. However, if we restrict our attention to *radial* functions on $\mathbb{H}^2$, meaning K -invariant functions such as the indicator $\chi_{B_r(0)}$ of a ball, then the restricted Fourier-Helgason-Laplace transform enjoys similar properties to the Euclidean Fourier-Laplace transform. To be more precise, is $f \in C_c(\mathbb{H}^2)$ is radial, again meaning $f(k.z) = f(z)$ for all $k \in K$, or equivalently $f(z) = f_o(d(z, o))$ for some $f_o : \mathbb{R}_{\geq 0} \rightarrow \mathbb{C}$, then

$$\widehat{f}(s) := \widehat{f}(s, \xi) = \int_{\mathbb{H}^2} f(z) \varphi_s(z) dm_{\mathbb{H}^2}(z),$$

where

$$\varphi_s(z) = \int_{\partial\mathbb{H}^2} \mathcal{P}(z, \xi)^s d\xi$$

is the *zonal spherical function* on $\mathbb{H}^2$ with parameter $s \in \mathbb{C}$ and $d\xi$ is the normalized Lebesgue measure on $\partial\mathbb{H}^2$ such that $\varphi_s(o) = 1$. We will typically consider φ_s as a function on $G = \mathrm{SL}_2(\mathbb{R})$, $\mathrm{SU}(1, 1)$ and write $\varphi_s(g)$ instead of $\varphi_s(g.o)$. By definition, φ_s is right- K -invariant and looking to the disk model we see that φ_s is also left- K -invariant,

$$\varphi_s(k_v g) = \frac{1}{2\pi} \int_0^{2\pi} \left(\frac{1 - |g.0|^2}{|g.0 - e^{i(v'-v)}|^2} \right)^s dv' = \varphi_s(g).$$

The transform $f \mapsto \widehat{f}$ for radial f has several names in the literature: *spherical transform*, *Mehler-Fock transform*, *Selberg transform* and *Harish-Chandra transform* are some names used.

3.2. Equivalent definitions of zonal spherical functions

Zonal spherical functions can be defined in several equivalent ways, and we mention them here. To state them we first prove the following Lemma.

Lemma 3.1. *The space $C_c(G, K)$ of bi- K -invariant functions forms an algebra under convolution and is commutative under this operation.*

Proof of commutativity. From the Cartan decomposition one sees that $Kg^{-1}K = KgK$ for every $g \in G$. Thus every $f \in C_c(G, K)$ satisfies $f(g^{-1}) = f(g)$ for all $g \in G$. Thus if $f_1, f_2 \in C_c(G, K)$ then

$$\begin{aligned} (f_1 * f_2)(g) &= \int_G f_1(h) f_2(h^{-1}g) dm_G(h) = \int_G f_1(gh) f_2(h^{-1}) dm_G(h) \\ &= \int_G f_1(h^{-1}g^{-1}) f_2(h) dm_G(h) = (f_2 * f_1)(g^{-1}) = (f_2 * f_1)(g). \end{aligned}$$

□

A pair of groups (G, K) for which the convolution algebra $C_c(G, K)$ is commutative is called a *Gelfand pair* and the associated quotient space $X = G/K$ is called a *commutative space*. Apart from our definition using the Poisson kernel, the zonal spherical functions φ_s are equivalently defined as continuous bi- K -invariant functions with $\varphi_s(o) = 1$ that are

- (1) eigenfunctions $-\Delta\varphi_s = s(1-s)\varphi_s$, (and more generally joint eigenfunctions for the algebra of G -invariant differential operators on $X = G/K$)
- (2) multiplicative linear functionals $\varphi_s : C_c(G, K) \rightarrow \mathbb{C}$, i.e.

$$(\widehat{f_1 * f_2})(s) = \widehat{f_1}(s)\widehat{f_2}(s).$$

- (3) diagonalizing convolution in the sense that

$$(\varphi_s * f)(g) = \widehat{f}(s)\varphi_s(g), \quad g \in G, \quad f \in C_c(G, K)$$

- (4) satisfying the functional equation

$$\varphi_s(g_1)\varphi_s(g_2) = \frac{1}{2\pi} \int_0^{2\pi} \varphi_s(g_1 k_v g_2) dv, \quad g_1, g_2 \in G.$$

Remark 3.2. If $G = KA^+K$ and $A \cong \mathbb{R}$ as in the case of $\mathbb{H}^2$, a criterion for (G, K) being Gelfand is that $Kg^{-1}K = KgK$ for all $g \in G$. In more general situations, for example the Gelfand pair $(\mathrm{SL}_n(\mathbb{R}), \mathrm{SO}(n))$ with $n \geq 3$, $A \cong \mathbb{R}^{n-1}$ and one checks that the property $Kg^{-1}K = KgK$ is not fulfilled for all $g \in \mathrm{SL}_n(\mathbb{R})$.

3.3. Positive-definite zonal spherical functions and irreducible spherical representations

We mention the special values of $s \in \mathbb{C}$ for which the zonal spherical functions are *positive-definite*, meaning

$$\langle f^* * f, \varphi_s \rangle \geq 0, \quad \text{for all } f \in C_c(G).$$

It is well-known that positive-definite functions are matrix coefficients of unitary representations of the reference group, and in the case of zonal spherical functions the positive-definite ones are the $s \in \mathbb{C}$ for which

$$\varphi_s(g) = \langle \pi_s(g)1, 1 \rangle_{\mathcal{H}_s}$$

where $\pi_s : G \rightarrow \mathcal{B}(\mathcal{H}_s)$ is a unitary irreducible (so called) K -spherical G -representation. The usefulness of the parameters $s \in \mathbb{C}$ for which φ_s is positive-definite is that they satisfy

$$(\widehat{f^* * f})(s) = |\widehat{f}(s)|^2, \quad f \in C_c(G, K).$$

The representations mentioned are given by the linear action

$$\pi_s(g)b(\xi) = P(g.o, \xi)^s b(g^{-1}.\xi)$$

for $b \in \mathcal{H}_s$, where the Hilbert space $\mathcal{H}_s$ is given in the disk model by

- (1) $\mathcal{H}_s = L^2(\partial\mathbb{D})$ for $s \in 1/2 + i\mathbb{R}$, (the *principal series*)
- (2) $\mathcal{H}_s = H^s(\partial\mathbb{D})$ for $s \in (0, 1)$ (the *complementary series*) with the inner product

$$\langle b_1, b_2 \rangle_{H^s} = \text{p.v.} \int \int \frac{b_1(\xi_1) \overline{b_2(\xi_2)}}{|\xi_1 - \xi_2|^s} d\xi_1 d\xi_2.$$

- (3) $\mathcal{H}_s = \mathbb{C}$ if $s = 0, 1$. (the *trivial representation*)

The irreducibility of these representations can be proven by analyzing actions of N, A, K on the boundary $\partial\mathbb{D}$ together with fairly basic Fourier analytic arguments, but showing that these are the only K -spherical irreducible representations is a highly non-trivial task. Moreover, there is an "explicit" isomorphism $(\pi_{1-s}, \mathcal{H}_{1-s}) \cong (\pi_s, \mathcal{H}_s)$ in terms of so called *Knapp-Stein kernels*, and one concludes that $\varphi_{1-s} = \varphi_s$ for all s mentioned. One can also, and maybe more easily, see that $\varphi_{1-s} = \varphi_s$ for all $s \in \mathbb{C}$ from the Helmholtz equation

$$-\Delta\varphi_s = s(1-s)\varphi_s.$$

Indeed, since φ_s defines a radial function on $\mathbb{H}^2$ this equation reduces to an ODE which has unique solutions. Since the eigenvalues $s(1-s)$ are fixed under the map $s \mapsto 1-s$ one concludes that $\varphi_{1-s} = \varphi_s$ from the initial condition $\varphi_s(o) = 1$.

4. SPECTRAL MEASURES

Let $\mathcal{S}^+ = [0, 1] \cup (1/2 + i\mathbb{R})$ be the set of s parameterizing the positive-definite zonal spherical functions for $\mathbb{H}^2$. Then we have the following analogue of the classical Bochner Theorem.

Theorem 4.1 (Spherical Bochner's Theorem). *Let $\phi \in C(G, K)$ be a continuous bi- K -invariant positive-definite function on G , meaning*

$$\langle f^* * f, \phi \rangle \geq 0, \quad \text{for all } f \in C_c(G).$$

Then there is a unique finite positive Borel measure σ_ϕ on $\mathcal{S}^+$ such that

$$\phi(g) = \int_{\mathcal{S}^+} \varphi_s(g) d\sigma_\phi(s).$$

In terms of representations, if $(\pi, \mathcal{H})$ is a unitary G -representation that is K -spherical in the sense that there is a vector $v \in \mathcal{H}$ such that $\pi(k)v = v$ for all $k \in K$, then the matrix coefficient $\langle \pi(g)v, v \rangle_{\mathcal{H}}$ is positive-definite and bi- K -invariant on G .

Thus there is a unique finite positive Borel measure σ_v on $\mathcal{S}^+$ such that

$$\langle \pi(g)v, v \rangle_{\mathcal{H}} = \int_{\mathcal{S}^+} \varphi_s(g) d\sigma_v(s).$$

In fact, all positive-definite $\phi \in C(G, K)$ arise in this way from the classical GNS-construction [Folland]. We have the following refinement of Bochner's Theorem for certain types of vectors coming from the algebra $C_c(G, K)$.

Theorem 4.2 (Godement, Björklund-B.). *Let $(\pi, \mathcal{H})$ be a unitary K -spherical G -representation and $\alpha : (\lambda_G, C_c(G)) \rightarrow (\pi, \mathcal{H})$ an equivariant map. Then there is a unique positive Radon measure σ_α such that*

$$\langle \pi(g)\alpha(f), \alpha(f) \rangle_{\mathcal{H}} = \int_{\mathcal{S}^+} \varphi_s(g) |\widehat{f}(s)|^2 d\sigma_\alpha(s), \quad f \in C_c(G, K).$$

Remark 4.3. The Theorem holds more generally for bounded measurable bi- K -invariant f with compact support, for instance when $f = \chi_{B_r(0)}$ is the indicator of a ball.

If $\mu \in \text{Prob}_G(\mathcal{M}_+(\mathbb{H}^2))$ is an invariant locally square-integrable random measure on $\mathbb{H}^2$ we take $\mathcal{H} = L_0^2(\mu)$ with the linear action

$$\pi(g) : \int_{\mathbb{H}^2} f(z) dp(z) \longmapsto \int_{\mathbb{H}^2} f(g^{-1} \cdot z) dp(z), \quad \int_{\mathbb{H}^2} f(z) dm_{\mathbb{H}^2}(z) = 0$$

and with the map

$$\alpha(f)(p) = \int_{\mathbb{H}^2} f(z) dp(z) - \rho_\mu \int_{\mathbb{H}^2} f(z) dm_{\mathbb{H}^2}(z), \quad f \in C_c(\mathbb{H}^2)$$

we get a unique positive Radon measure σ_μ on $\mathcal{S}^+$ such that

$$\int \left| \int_{\mathbb{H}^2} f(z) dp(z) - \rho_\mu \int_{\mathbb{H}^2} f(z) dm_{\mathbb{H}^2}(z) \right|^2 d\mu(p) = \int_{\mathcal{S}^+} \varphi_s(g) |\widehat{f}(s)|^2 d\sigma_\mu(s)$$

for all radial $f \in C_c(\mathbb{H}^2)$. As in the Euclidean case, if μ is ergodic then $\sigma(\{0, 1\}) = 0$.

In particular, the L^2 -discrepancy of μ over a compact set $B \subset \mathbb{H}^2$ is given by

$$\mathcal{D}_B^{L^2}(\mu) = \left(\int \left| p(B) - \rho_\mu |B| \right|^2 d\mu(p) \right)^{1/2}$$

and in the case of $B = B_r(0)$ it satisfies

$$\mathcal{D}_{B_r(0)}^{L^2}(\mu)^2 = \|\alpha(\chi_{B_r(0)})\|_{L_0^2(\mu)}^2 = \int_{\mathcal{S}^+} |\widehat{\chi}_{B_r(0)}(s)|^2 d\sigma_\mu(s).$$

What remains in order to state Beck-type Theorems for $\mathbb{H}^2$ is the asymptotics of $\widehat{\chi}_{B_r(0)}$ as $r \rightarrow +\infty$. This will constitute most of the remainder of the notes.

Remark 4.4. There is an alternative way of proving the existence of the spectral measure σ_μ as mentioned above: There is a *horocyclic Radon transform* $\mathcal{R} : C_c(\mathbb{H}^2) \rightarrow C_c(\mathbb{R} \times \mathbb{S}^1)$ on $\mathbb{H}^2$, which in the upper-half plane model is given by

$$\mathcal{R}f(t, v) = e^{t/2} \int_{\mathbb{R}} f(k_v a_t n_s . i) ds = e^{t/2} \int_{\mathbb{R}} f\left(\frac{\cos(v)e^t(i+s) - \sin(v)}{\sin(v)e^t(i+s) + \cos(v)}\right) ds.$$

The interpretation is that $\mathcal{R}f$ is a function on the space of horocycles in $\mathbb{H}^2$, all of which are given by $s \mapsto k_v a_t n_s . i$ for some $t \in \mathbb{R}, v \in \mathbb{S}^1$, and the normalizing constant $e^{t/2}$ arises from the group $AN < G$ being non-unimodular (a phenomenon that does not appear in the Euclidean setting). Restricting to radial f gives rise to the *Abel transform* $\mathcal{A} : C_c(\mathbb{H}^2)^{\text{rad}} \rightarrow C_c(\mathbb{R})^{\text{even}}$

$$\mathcal{A}f(t) = e^{t/2} \int_{\mathbb{R}} f(a_t n_s . i) ds = e^{t/2} \int_{\mathbb{R}} f(e^t(i+s)) ds.$$

A remarkable property of this Abel transform is that it takes convolution on G to convolution on $\mathbb{R}$,

$$\mathcal{A}(f_1 * f_2) = \mathcal{A}f_1 *_\mathbb{R} \mathcal{A}f_2, \quad f_1, f_2 \in C_c(G, K) \cong C_c(\mathbb{H}^2)^{\text{rad}}.$$

In particular, $\mathcal{A}$ preserves positive-definiteness. Moreover, the Abel transform relates the spherical transform on $\mathbb{H}^2$ to the ordinary Fourier transform on $A \cong \mathbb{R}$ via the *Fourier slice Theorem*, sometimes called the "*holy trinity*",

$$\widehat{f}(s) = \int_{\mathbb{R}} \mathcal{A}f(t) e^{-2\pi i s t} dt.$$

Using this machinery one can use the Bochner-Schwartz theorem on $\mathbb{R}$ to find a positive Radon measure σ_α on $\mathbb{C}$ such that

$$\langle \pi(g)\alpha(f), \alpha(f) \rangle_{\mathcal{H}} = \int_{\mathbb{C}} \varphi_s(g) \widehat{f}(s) \overline{\widehat{f}(\bar{s})} d\sigma_\alpha(s), \quad f \in C_c(G, K).$$

However, it is not clear from this perspective that σ_α is supported on the subset $\mathcal{S}^+$ of parameters for which φ_s is positive-definite, which is crucial since we will need positivity of

$$\int_{\mathbb{C}} \widehat{\chi}_{B_r(0)}(s) \overline{\widehat{\chi}_{B_r(0)}(\bar{s})} d\sigma_\alpha(s)$$

for the hyperbolic analogue of Beck's Theorem.

5. ASYMPTOTICS FOR THE SPHERICAL TRANSFORM OF A BALL

5.1. Computing the zonal spherical functions

Since φ_s is a bi- K -invariant function on $G = KA^+K$, it suffices to compute $\varphi_s(a_t)$ for $t \geq 0$. In the unit disk model, we get

$$\begin{aligned}\varphi_s(a_t) &= \frac{1}{2\pi} \int_0^{2\pi} \left(\frac{1 - \tanh^2(t/2)}{|\tanh(t/2) - e^{iv}|^2} \right)^s dv = \frac{1}{2\pi} \int_0^{2\pi} |\sinh(t/2) - \cosh(t/2)e^{iv}|^{-2s} dv \\ &= \frac{1}{2\pi} \int_0^{2\pi} (\cosh(t) - \sinh(t) \cos(v))^{-s} dv \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} (\cosh(t) + \sinh(t) \cos(v))^{-s} dv = P_{-s}(\cosh(t)),\end{aligned}$$

where P_{-s} is the *Legendre function*

$$P_{-s}(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} (z + (z^2 - 1)^{1/2} \cos(v))^{-s} dv.$$

In total we get

$$\varphi_s(g) = P_{-s}(\cosh(d(g.o, o)))$$

for all $g \in G$. This is surprisingly not very useful as it is hard to find asymptotic expansions of P_{-s} in the literature. The following substitution will turn out to be extremely important: Let $u = \tan(v/2)$ so that $dv = 2(1 + u^2)^{-1} du$, which gives us

$$\begin{aligned}\varphi_s(a_t) &= \frac{1}{\pi} \int_{\mathbb{R}} \left(\cosh(t) + \sinh(t) \frac{1 - u^2}{1 + u^2} \right)^{-s} \frac{du}{1 + u^2} \\ &= \frac{1}{\pi} \int_{\mathbb{R}} ((1 + u^2) \cosh(t) + (1 - u^2) \sinh(t))^{-s} (1 + u^2)^{s-1} du.\end{aligned}$$

A short calculation yields

$$(1 + u^2) \cosh(t) + (1 - u^2) \sinh(t) = e^t (1 + u^2 e^{-2t}),$$

and we find that

$$\varphi_s(a_t) = \frac{e^{-ts}}{\pi} \int_{\mathbb{R}} (1 + u^2 e^{-2t})^{-s} (1 + u^2)^{s-1} du.$$

All of a sudden we find that φ_s decays at infinity whenever $0 < \operatorname{Re}(s) < 1$, which implies the so called Howe-Moore property of G for the K -spherical unitary irreducible representations. In fact, the asymptotics are closely related to a central object in non-Euclidean harmonic analysis.

Lemma 5.1. *Suppose $0 < \operatorname{Re}(s) < 1$. Then*

$$\lim_{t \rightarrow +\infty} e^{ts} \varphi_s(a_t) = \int_{\mathbb{R}} (1 + u^2)^{s-1} du = \frac{1}{\sqrt{\pi}} \frac{\Gamma(\frac{1}{2} - s)}{\Gamma(1 - s)}.$$

Setting $s = 1/2 - i\lambda$ gives us the *Harish-Chandra c-function*

$$c(\lambda) = \frac{1}{\sqrt{\pi}} \frac{\Gamma(i\lambda)}{\Gamma(\frac{1}{2} + i\lambda)},$$

which is at the heart of harmonic analysis on Lie groups. Using properties of the Gamma function one has

$$|c(\lambda)|^{-2} = \pi\lambda \tanh(\pi\lambda)$$

which shows up in the *Plancherel formula* for $\mathbb{H}^2$,

$$\int_{\mathbb{H}^2} |f(z)|^2 dm_{\mathbb{H}^2}(z) = \int_{\mathbb{R}} |\widehat{f}(\frac{1}{2} + i\lambda)|^2 \pi\lambda \tanh(\pi\lambda) d\lambda.$$

5.2. Large-radius asymptotics for the spherical transform of a ball

Lemma 5.2. *As $r \rightarrow +\infty$ we have the following asymptotics:*

(1) *for $s = \frac{1}{2} - i\lambda$ with $\lambda \in \mathbb{R} \setminus \{0\}$,*

$$\lim_{R \rightarrow +\infty} \frac{1}{R} \int_0^R \frac{|\widehat{\chi}_{B_r(0)}(\frac{1}{2} - i\lambda)|^2}{\cosh(r)} dr = \frac{|c(\lambda)|^2}{\frac{1}{4} + \lambda^2} > 0.$$

(2) *for $s = \frac{1}{2}$,*

$$\lim_{r \rightarrow +\infty} \frac{|\widehat{\chi}_{B_r(0)}(\frac{1}{2})|^2}{r^2 \cosh(r)} = \text{const.} > 0$$

(3) *for $s \in [0, 1] \setminus \{\frac{1}{2}\}$,*

$$\lim_{r \rightarrow +\infty} \frac{|\widehat{\chi}_{B_r(0)}(s)|^2}{\cosh(r)^{1+|2s-1|}} = \frac{|c(i(s - \frac{1}{2}))|^2}{s(1-s)} > 0.$$

Proof sketch of (1). By Lemma 5.1 we have that

$$\begin{aligned} \frac{|\widehat{\chi}_{B_r(0)}(\frac{1}{2} - i\lambda)|^2}{\cosh(r)} &= \frac{1}{\cosh(r)} \left| \int_0^r \varphi_{\frac{1}{2}-i\lambda}(t) 2 \sinh(t) dt \right|^2 \\ &\sim \frac{|c(\lambda)|^2}{\cosh(r)} \left| \int_0^r e^{(\frac{1}{2}-i\lambda)t} dt \right|^2 \\ &= \frac{|c(\lambda)|^2 e^r}{(\frac{1}{4} + \lambda^2) \cosh(r)} |e^{-i\lambda r} - e^{-r/2}|^2 \\ &= \frac{|c(\lambda)|^2 e^r}{(\frac{1}{4} + \lambda^2) \cosh(r)} (\cos^2(\lambda r) - 2e^{-r/2} \cos(\lambda r) + e^{-r}) \end{aligned}$$

Asymptotic averaging over r gives the desired result. □

6. HYPERBOLIC BECK'S THEOREM

The same approach as in the Proof sketch of Theorem 1.2, now using the existence of the spectral measure σ_μ on the Fourier domain $\mathcal{S}^+ = [0, 1] \cup (\frac{1}{2} + i\mathbb{R})$ and Lemma 5.2 gives us the following analogue of Beck's Theorem on $\mathbb{H}^2$.

Theorem 6.1 (Björklund-B.). *Let μ be an $\mathrm{SU}(1, 1)$ -invariant (ergodic) locally square-integrable random measure on $\mathbb{D}$. Then*

$$\limsup_{r \rightarrow +\infty} \frac{\mathcal{D}_{B_r(0)}^{L^2}(\mu)}{\cosh(r)^{\frac{1}{2}}} > 0.$$

Moreover,

(1) *if $\sigma_\mu(\{\frac{1}{2}\}) > 0$, then*

$$\liminf_{r \rightarrow +\infty} \frac{\mathcal{D}_{B_r(0)}^{L^2}(\mu)}{r \cosh(r)^{\frac{1}{2}}} > 0,$$

(2) *if $\sigma_\mu((\frac{1+\delta}{2}, 1)) > 0$ for some $0 < \delta < 1$ then*

$$\liminf_{r \rightarrow +\infty} \frac{\mathcal{D}_{B_r(0)}^{L^2}(\mu)}{\cosh(r)^{\frac{1+\delta}{2}}} > 0.$$

Remark 6.2. We make the following remarks:

- Since $\mathrm{SU}(1, 1)$ is non-amenable, it is not always the case that a compact hull Ω_P admits an invariant measure. In the case of model sets however, it does due to work of Björklund-Hartnick-Pogorzelski in [2, Theorem 1.1]. The spectral measure and, even its support is however unknown. The support is conjectured to lie in the principal series $\frac{1}{2} + i\mathbb{R}$ (but is probably extremely hard to prove).
- There are lattices $\Gamma < \mathrm{SU}(1, 1)$, i.e. discrete subgroups of finite covolume, such that (1) and (2) are fulfilled for some δ . Moreover, Jenni in [4, Section 1.2] has constructed a cocompact lattice $\Gamma < \mathrm{SU}(1, 1)$ for which the spectral measure (which is atomic) has support in $\frac{1}{2} + i\mathbb{R} \setminus \{0\}$, consequently satisfying

$$0 < \limsup_{r \rightarrow +\infty} \frac{\mathcal{D}_{B_r(0)}^{L^2}(\Gamma)}{\cosh(r)^{\frac{1}{2}}} < +\infty.$$

- To our knowledge, it's not known whether there is a point process μ satisfying the analogue of the Parnowski-Sobolev result

$$\liminf_{r \rightarrow +\infty} \frac{\mathcal{D}_{B_r(0)}^{L^2}(\mu)}{\cosh(r)^{\frac{1}{2}}} = 0.$$

- For each $0 < \delta < 1$ there is a so called Gaussian analytic function whose zero set μ_δ satisfies (2) in the above Theorem, a result due to Buckley in [3, Theorem 1(c)]

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